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Superintegrable systems on the loop algebras

A V Tsiganov

Department of Mathematical and Computational Physics, Institute of Physics, St Petersburg University, 198904 St Petersburg, Russia

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Abstract. In this work we consider superintegrable systems in the classical R-matrix method. By using outer automorphisms of the loop algebras we construct new superintegrable systems with rational potentials from geodesic motion on $\mathbb{R}^{2n}$.

1. Introduction

We shall consider classical integrable Hamiltonian systems on the coadjoint orbits of finite-dimensional Lie algebras according to [1]. The dual space $\mathfrak{g}^*$ to the Lie algebra $\mathfrak{g}$ is equipped with the natural Lie–Poisson brackets specified by the condition that the Poisson bracket of two linear functions on $\mathfrak{g}^*$ coincides with their Lie bracket in $\mathfrak{g}$. Let H be a function on $\mathfrak{g}^*$, $\nabla H \in \mathfrak{g}$ the gradient of H. In the space $C^\infty(\mathfrak{g}^*)$ of smooth functions, H determines the evolution with the associated Hamiltonian equation

$$\dot{x} = -(\mathrm{ad}^*_{\nabla H}) \cdot x \qquad x \in \mathfrak{g}^*. \tag{1.1}$$

If $\mathfrak{g}$ is self-dual, i.e. has a nondegenerate inner product which allows us to identify $\mathfrak{g}^*$ with $\mathfrak{g}$ and ad^* with ad, then (1.1) takes on the usual form

$$\dot{x} = \{H, x\} = -(\mathrm{ad}_{\nabla H}) \cdot x \qquad x \in \mathfrak{g}. \tag{1.2}$$

Henceforth we shall always identify $\mathfrak{g}^*$ with $\mathfrak{g}$ and ad^* with ad.

Function $I \in C^\infty(\mathfrak{g})$ is called an integral of evolution with a Hamiltonian H if

$$\{H, I\} = 0.$$

The evolution on a $2n$-dimensional symplectic manifold M with the Hamiltonian H is called completely integrable if there exists n functions $I_1, \ldots, I_n$, which are independent integrals in the involution for the Hamiltonian H

$$\{I_i, I_j\} = 0 \qquad i, j \leqslant n. \tag{1.3}$$

The functions $I_1, \ldots, I_n$ are independent, if forms $\mathrm{d}I_1, \ldots, \mathrm{d}I_n$ are linearly independent on the common level surfaces of these functions.

The evolution on a manifold M, $\dim M = 2n$ with the Hamiltonian H is called superintegrable or degenerate, if there exists more than n independent integrals of motion $\{I_j\}_{j=1}^k$, $k > n$ and n of which are in the involution (1.3) [1, 17, 24]. For the superintegrable systems all the integrals $\{I_j\}_{j=1}^k$, $k > n$ are generators of the polynomial associative algebra, whose defining relations are polynomials of certain order in generators (see [4, 5] for a

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collection of original papers). The main example of the superintegrable systems is a free motion with the following Hamiltonian and equations of motion:

$$H = \sum_{j=1}^{n} p_j^2 \qquad \dot{q}_j = p_j \qquad \dot{p}_j = 0 \tag{1.4}$$

here p_j, q_j are canonical variables on $M = \mathbb{R}^{2n}$. Integrals of motion in the involution and additional integrals of motion may be defined as

$$I_k = p_k^2 \qquad J_{ik} = p_i q_k - q_i p_k. \tag{1.5}$$

Other known classical superintegrable systems with arbitrary numbers of degrees of freedom are the harmonic oscillator, the Kepler problem and the Calogero system [20]. Note that the Kepler problem and the Calogero model may be obtained from the geodesic motion (1.4) on spaces of constant curvature [20]. Other examples of superintegrable systems can be constructed by using either purely algebraic techniques [1, 4, 5, 17] or the separation of variables method at $n = 2, 3$ [24, 8, 5]. Some individual examples of superintegrable systems are listed in [20] with the corresponding references.

Our aim is to show how superintegrable systems fit into a general pattern based on the notion of the classical R-matrix [10, 21]. The main advantage of such embedding is that important structure elements for superintegrable systems, such as a Lax representation, separation of variables [5] and spectrum-generating algebra (dynamical algebra) [4, 18] can be systematically derived from the underlying standard R-matrix formalism, which is a prerequisite for the study of the quantum case.

As an example, let us consider the degenerate geodesic motion (1.4)–(1.5). Introduce an arbitrary set of points $\{\delta_k\}_{k=1}^n$ and define some functions $p(\lambda), x(\lambda)$ with the following property

$$p(\lambda)|_{\lambda=\delta_k} = p_k \qquad x(\lambda)|_{\lambda=\delta_k} = x_k.$$

The explicit form of the functions is not very important here. By using these functions the involutive family of integrals I_k (1.5) may be generated by a one-variable function $I(\lambda)$ whereas the second set of integrals J_{ik} (1.5) can be recovered from the two-variable $J(\lambda, \mu)$ function only

$$I_k = I(\lambda)|_{\lambda=\delta_k} = p^2(\lambda)|_{\lambda=\delta_k}$$
$$J_{ik} = J(\lambda, \mu)|_{\lambda=\delta_i \mu=\delta_k} = [p(\lambda)x(\mu) - x(\lambda)p(\mu)]_{\lambda=\delta_i \mu=\delta_k}.$$

Keeping this analogy in mind, to consider superintegrable systems on the loop algebras $\mathcal{L}(\mathfrak{g}, \lambda)$, we introduce the multivariable universal enveloping algebras of $\mathcal{L}(\mathfrak{g}, \lambda)$. In this case the complete set of noncommutative integrals of motion is defined on the special algebraic surfaces.

We propose a dressing procedure allowing us to construct the new superintegrable systems starting from known ones. As a natural initial point we shall select a geodesic motion (1.4) on the Riemannian spaces of constant curvature. To construct the new Lax equations associated with a potential superintegrable motion we apply the outer automorphism of the underlying algebra $\mathfrak{g}$ [25] directly to the Lax equations associated with a geodesic motion.

The paper is organized as follows. In section 2 we briefly recall the notion of the classical R-matrix method. In section 3 superintegrable systems are constructed in R-matrix formalism, while section 4 contains some examples.

2. Method of the classical *R*-matrix

A systematic way for realizing an integrable Hamiltonian system on coadjoint orbits of the Lie algebras is provided by the R-matrix method [10, 21].

Recall that the classical R-matrix on a Lie algebra $\mathfrak{g}$ is a linear operator $R \in \mathrm{End}(\mathfrak{g})$ such that the bracket on $\mathfrak{g}$

$$[X, Y]_R = \tfrac{1}{2}([RX, Y] + [X, RY]) \qquad X, Y \in \mathfrak{g} \tag{2.1}$$

satisfies the Jacobi identity [21, 22]. In this case there are two structures of a Lie algebra in the linear space $\mathfrak{g}^*$ given by the original Lie bracket and the R-bracket (2.1), respectively. Definition of the second algebraic structure associated with R-brackets is motivated by the following result. Let $\mathfrak{I}(\mathfrak{g})$ be the ring of Casimir functions on $\mathfrak{g}^*$. Functions $\tau_j \in \mathfrak{I}(\mathfrak{g})$ are invariants with respect to the original Lie structure and they are in the involution with respect to the R-bracket on $\mathfrak{g}^*$. Namely, if τ is an invariant of the coadjoint action of $\mathfrak{g}$, the associated Hamiltonian equation (1.1) on $\mathfrak{g}^*$ is equal to

$$\frac{\mathrm{d}L}{\mathrm{d}t} = -\mathrm{ad}^*_A \cdot L \qquad A = \tfrac{1}{2}R(\mathrm{d}\tau(L)) \qquad L \in \mathfrak{g}^*. \tag{2.2}$$

If $\mathfrak{g}$ is self-dual, so that $\mathrm{ad}^* = \mathrm{ad}$, then equation (2.2) takes on the usual Lax form [21]

$$\frac{\mathrm{d}L}{\mathrm{d}t} = [L, A]$$

and the R-bracket (2.1) can be rewritten in tensor form [10, 21, 22].

It is obvious, that for any R-matrix in (2.2) all the Casimir functions τ_j on $\mathfrak{g}^*$ give rise to integrals of motion in the involution [10, 21]. Our aim is to find the origin of an appearance of the special superintegrable Hamiltonians and their additional integrals of motion in this method.

Finite-dimensional, simple Lie algebras $\mathfrak{g}$ lead to generalized Toda lattices and other mechanical systems that may be integrated in elementary functions [10, 21, 22]. This excludes the most interesting examples of integrable systems which come from analytical mechanics, since their solutions are Abelian functions of the time variable. In order to obtain dynamical systems of this type, one has to consider the class of the so-called affine Lie algebras, or loop algebras $\mathcal{L}(\mathfrak{g}, \lambda)$ [21]. Recall, that loop algebra $\mathcal{L}(\mathfrak{g}, \lambda)$ can be realized as an algebra of the Laurent polynomials with coefficients in $\mathfrak{g}$

$$\mathcal{L}(\mathfrak{g}, \lambda) = \mathfrak{g}[\lambda, \lambda^{-1}] = \left\{ x(\lambda) = \sum_i x\lambda^i,\ x \in \mathfrak{g} \right\}$$

and with the commutator $[x\lambda^i, y\lambda^j] = [x, y]\lambda^{i+j}$.

Let $L(\lambda) \in \mathcal{L}(\mathfrak{g}, \lambda)$ be a generic point in the loop algebra, which is regarded as a Lax matrix. The tensor form [10, 21, 22] of the corresponding R-bracket is given by

$$\begin{aligned} &\{L_1(\lambda), L_2(\mu)\} = [r_{12}(\lambda, \mu), L_1(\lambda)] - [r_{21}(\lambda, \mu), L_2(\mu)] \\ &r_{21}(\lambda, \mu) = P_{12} r_{12}(\lambda, \mu) P_{12} \end{aligned} \tag{2.3}$$

where P_{12} is a permutation operator in $\mathcal{L}(\mathfrak{g}, \lambda) \otimes \mathcal{L}(\mathfrak{g}, \mu)$ and $r_{ij}(\lambda, \mu)$ are kernels of the corresponding operators R and R^* in (2.1) [22]. Note that the R-matrix scheme is extended easily to the twisted subalgebras of loop algebra $\mathcal{L}(\mathfrak{g}, \lambda)$ and the corresponding matrices r_{12} have rational, trigonometric and elliptic dependencies on spectral parameters.

According to an algebra homomorphism [12]

$$U(\mathcal{L}(\mathfrak{g}, \lambda)) \rightarrow \mathbb{C}[\lambda, \lambda^{-1}] \otimes U(\mathfrak{g})$$

we can construct the one-variable universal enveloping algebra $U(\mathcal{L}(\mathfrak{g},\lambda)$. In this case the Casimir functions on the loop algebras can be recovered by the ad-invariants $\tau_j(x)$ in $\mathfrak{I}(\mathfrak{g})$

$$\tau_{j,\phi} = \mathrm{Res}|_{\lambda=0}\phi(\lambda)\cdot\tau_j(x(\lambda)) \qquad \tau_j(x)\in\mathfrak{I}(\mathfrak{g}) \qquad \phi(\lambda)\in\mathbb{C}[\lambda,\lambda^{-1}] \tag{2.4}$$

where $\phi(\lambda)$ is some rational function of the spectral parameter λ with numerical values [21].

It is obvious, that an application of the ad-invariant functions τ_j is a basic tool in the R-matrix method [10, 21] and, therefore, we consider these functions in greater detail to assume the standard identification of the dual spaces.

To begin with let us recall some necessary facts from the notion of a universal enveloping algebra [6]. Let $\mathfrak{g}$ be a Lie algebra and $T(\mathfrak{g})$ be the tensor algebra of the vector space $\mathfrak{g}$

$$T = T^0\oplus T^1\oplus T^2\ldots \qquad T^n = \mathfrak{g}\otimes\mathfrak{g}\otimes\cdots\otimes\mathfrak{g} \qquad n \text{ times}. \tag{2.5}$$

If J is the two-sided ideal of T generated by the tensors

$$x\otimes y - y\otimes x - [x,y] \qquad x,y\in\mathfrak{g}$$

then the associative algebra T/J is called the universal enveloping algebra, which is usually denoted by $U(\mathfrak{g})$.

Let $m\geqslant 0$ be an integer. The vector subspace of $U(\mathfrak{g})$ generated by the products $x_1x_2\ldots x_j$, where $x_1,x_2,\ldots,x_j\in\mathfrak{g}$ and $j\leqslant m$ is denoted by $U_m(\mathfrak{g})$. We have

$$U_0(\mathfrak{g}) = \mathbb{C}\cdot 1 \qquad U_1(\mathfrak{g}) = \mathbb{C}\cdot 1\oplus\mathfrak{g} \qquad U_i(\mathfrak{g})U_j(\mathfrak{g})\subset U_{i+j}(\mathfrak{g}).$$

This sequence is the canonical filtration of $U(\mathfrak{g})$.

According to the Birkhoff–Witt theorem $T(\mathfrak{g}) = J\oplus S(\mathfrak{g})$ and algebra $U(\mathfrak{g})$ is isomorphic to the symmetric algebra $S(\mathfrak{g})$ as a vector space. If $x_1,x_2,\ldots,x_m\in\mathfrak{g}$, then

$$w(x_1x_2\ldots x_m) = \frac{1}{m!}\sum_{\pi}P_\pi x_1x_2\ldots x_m = \frac{1}{m!}\sum_{\pi}x_{\pi(1)}x_{\pi(2)}\ldots x_{\pi(m)} \tag{2.6}$$

here P_π means the permutation operator corresponding to a certain Young diagram π [6]. The map w (2.6) is a bijection of $S(\mathfrak{g})$ onto $U(\mathfrak{g})$, which is called the symmetrization.

The Casimir functions or the ad-invariant functions on $\mathfrak{g}$ form a centre $\mathfrak{I}(\mathfrak{g})$ of $U(\mathfrak{g})$. The symmetrization mapping (2.6) allows us to construct $\mathfrak{I}(\mathfrak{g})$ by using the ad-invariants of commutative algebra $S(\mathfrak{g})$ [6].

Studying the superintegrable systems we have to introduce the multivariable tensor algebra

$$\begin{aligned}&T(\mathfrak{g},\lambda,\mu,\ldots) = T^0\oplus T^1\oplus T^2\ldots\\ &T^m(\mathfrak{g},\lambda,\mu,\ldots,\nu) = \mathcal{L}(\mathfrak{g},\lambda)\otimes\mathcal{L}(\mathfrak{g},\mu)\otimes\cdots\otimes\mathcal{L}(\mathfrak{g},\nu) \qquad m \text{ times}\end{aligned} \tag{2.7}$$

and canonical filtration of the corresponding enveloping algebra $U(\mathfrak{g},\lambda,\mu,\ldots)$ generated by subspaces $U_m(\mathfrak{g},\lambda,\mu,\ldots,\nu)$. These vector subspaces are produced by subspaces $U_m(\mathfrak{g})$

$$x_{i_1i_2\ldots i_k}(\lambda,\mu,\ldots,\nu) = \sum_{j_1,j_2,\ldots,j_k}x_{i_1i_2\ldots i_k}\lambda^{j_1}\mu^{j_2}\ldots\nu^{j_k} \qquad k\leqslant m \qquad x_{i_1i_2\ldots i_k}\in U_m(\mathfrak{g}).$$

In just the same way as for a Lie algebra $\mathfrak{g}$ [6], one defines the canonical mapping of the loop algebra $\mathcal{L}(\mathfrak{g},\lambda)$ into $U(\mathfrak{g},\lambda,\mu,\ldots)$. Any element $L(\lambda)$ of $\mathcal{L}(\mathfrak{g},\lambda)$ can be embedded into $U_m(\mathfrak{g},\lambda,\mu,\ldots,\nu)$

$$L_j(\lambda_j) = \mathrm{id}_1\otimes\cdots\otimes\mathrm{id}_{j-1}\otimes L(\lambda_j)\otimes\mathrm{id}_{j+1}\otimes\cdots\otimes\mathrm{id}_m\in U_m(\mathfrak{g},\lambda,\mu,\ldots,\nu) \tag{2.8}$$

and

$$L^{(k)}_{j_1 j_2 \ldots j_k}(\lambda_1, \lambda_2, \ldots, \lambda_k) = \prod_{n=1}^{k} L_{j_n}(\lambda_{j_n}) \qquad 1 \leqslant k \leqslant m \tag{2.9}$$

$$L^{(m)}(\lambda, \mu, \ldots, \nu) = L(\lambda) \otimes L(\mu) \otimes \cdots \otimes L(\nu) \qquad m \text{ times}. \tag{2.10}$$

Here $\lambda_1 = \lambda, \lambda_2 = \mu, \ldots \lambda_m = \nu$.

The Poisson brackets between the elements $L^{(k)}_{j_1 j_2 \ldots j_k}(\lambda, \mu, \ldots, \nu)$ (2.9) can be written in the 'generalized' R-matrix form. For instance

$$\begin{aligned}\{L_{12}(\lambda, \mu), L_3(\nu)\} = {}& [r_{13}(\lambda, \nu) + r_{23}(\mu, \nu), L_{12}(\lambda, \mu)] - [r_{31}(\lambda, \nu), L_{23}(\mu, \nu)] \\ & - [r_{23}(\mu, \nu), L_{13}(\lambda, \nu)]\end{aligned} \tag{2.11}$$

where $r_{ij}(\lambda_i, \lambda_j)$ are R-matrices, which act nontrivially in the corresponding subspaces of $T_m(\mathfrak{g}, \lambda, \mu, \ldots, \nu)$ (2.7) and

$$\begin{aligned}\{L_{12}(\lambda, \mu), L_{34}(\nu, \eta)\} = {}& [r^{(1)}(\lambda, \mu, \nu, \eta), L_1(\lambda)] + [r^{(2)}(\lambda, \mu, \nu, \eta), L_2(\mu)] \\ & - [r^{(3)}(\lambda, \mu, \nu, \eta), L_3(\nu)] - [r^{(4)}(\lambda, \mu, \nu, \eta), L_4(\eta)]\end{aligned} \tag{2.12}$$

where

$$\begin{aligned}
r^{(1)}(\lambda, \mu, \nu, \eta) &= \tilde{r}(\lambda, \mu, \nu, \eta) + P_{34}\tilde{r}(\lambda, \mu, \eta, \nu)P_{34} \\
r^{(2)}(\lambda, \mu, \nu, \eta) &= P_{12}r^{(1)}(\mu, \lambda, \nu, \eta)P_{12} \\
r^{(3)}(\lambda, \mu, \nu, \eta) &= P_{13}[\tilde{r}(\lambda, \mu, \nu, \eta) + P_{12}\tilde{r}(\mu, \lambda, \nu, \eta)P_{12}]P_{13} \\
r^{(4)}(\lambda, \mu, \nu, \eta) &= P_{34}r^{(3)}(\lambda, \mu, \eta, \nu)P_{34} \\
\tilde{r}(\lambda, \mu, \nu, \eta) &= r_{13}(\lambda, \nu)L_{24}(\mu, \eta).
\end{aligned}$$

Here P_{ij} are operators of pairwise permutations in the tensor algebra $T_m(\mathfrak{g}, \lambda, \mu, \ldots, \nu)$ (2.7).

Integrals of motion in the involution are completely defined by the ad-invariants (2.4) of $\mathcal{L}(\mathfrak{g}, \lambda)$ and by the linear R-bracket (2.3) [10, 21]. Description of the superintegrable systems requires us to consider ad-invariants of the multivariable algebra $U(\mathfrak{g}, \lambda, \mu, \ldots)$, more complicated embedding $L^{(k)}_{j_1 j_2 \ldots j_k}(\lambda, \mu, \ldots, \nu)$ (2.9) and (2.10) and analogue of the symmetrization mapping.

Theorem 1. If the second Lax matrix $A(\lambda)$ is independent on spectral parameter λ, then the following elements of the multivariable algebra $U(\mathfrak{g}, \lambda, \mu, \ldots)$

$$L^{(k,\pi)}(\lambda, \mu, \ldots, \nu) = P_\pi L^{(k)}_{12\ldots k}(\lambda_1, \lambda_2, \ldots, \lambda_k) = L^{(k)}_{\pi(1)\pi(2)\ldots\pi(k)}(\lambda_{\pi(1)}, \lambda_{\pi(2)}, \ldots, \lambda_{\pi(k)}) \tag{2.13}$$

give rise to integrals of motion. Here P_π means the permutation operator corresponding to a certain Young diagram π.

Since $A(\lambda)$ is independent on λ we have that the equation of motion for the elements $L^{(k,\pi)}(\lambda, \mu, \ldots, \nu)$ has a Lax form

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} L^{(k,\pi)}(\lambda, \mu \ldots, \nu) &= [L^{(k,\pi)} A^{(k)}] \\
A^{(k)} \equiv \sum_{j=1}^{k} A_j(\lambda_j) &= \sum_{j=1}^{k} A_{\pi(j)}(\lambda_{\pi(j)}).
\end{aligned}$$

Of course, to construct the degenerate systems we have to prove that one can obtain the sufficient number of the functionally independent integrals of motion from the elements

(2.13). We have to solve this problem for any given superintegrable system on the loop algebras. It may be quite difficult to decide which Lie algebra is associated with a given mechanical system. As usual, the way around this difficulty is to reverse the reasoning and to study all possible dynamical systems associated with a given Lie algebra. Therefore, in the next section, we consider degenerate systems associated with the $sl(n)$ algebra.

3. Superintegrable systems on $sl(n)$

As an application to the integrable systems let us suppose, for concreteness, that every element $L(\lambda)$ of the loop algebra $\mathcal{L}(\mathfrak{g},\lambda)$ be a matrix in some fixed matrix representation of the algebra $\mathfrak{g}$ and the representation space be an auxiliary space [10]. Now we study an algebra $\mathfrak{g}=sl(n,\mathbb{C})$ in the fundamental representation and begin with the standard rational R-matrix [10, 21]. In this case let the element $L(\lambda)$ of $\mathcal{L}(\mathfrak{g},\lambda)$ be a $n\times n$ matrix in the auxiliary space $\mathbb{C}^n$ and the matrix elements of $L(\lambda)$ be some rational functions of spectral parameter λ.

The basis of invariant functions in $\mathfrak{I}(sl(n,\mathbb{C}))$ could be selected as

$$\tau_k(L(\lambda))=\frac{1}{k}\operatorname{tr}L^k(\lambda)\qquad k\leqslant n. \tag{3.1}$$

The family of the integrals of motion in the involution is generated by $\tau_k(L(\lambda))$ [10, 21]

$$I_{i,k}(L)=\Phi_\lambda^{(i)}(\tau_k(\lambda)) \tag{3.2}$$

where $\Phi_\lambda^{(i)}$ are various linear functionals defining a set of the functionally independent integrals of evolution. For instance,

$$\Phi_\lambda^{(i)}(z)=\operatorname{Res}|_{\lambda=0}(\phi_i(\lambda)\cdot z)\qquad \phi_i(\lambda)\in\mathbb{C}[\lambda,\lambda^{-1}] \tag{3.3}$$

here $\phi_i(\lambda)$ are some functions of spectral parameter with numerical values. In this case integrals $I_{i,k}$ are at most k-order polynomials in generators $\mathfrak{g}$ and $I_{i,k}\in U_k(\mathfrak{g})$.

The Lax equation (2.2) associated with the Hamiltonian $I_{i,k}$ (3.2) is equal to

$$\frac{\mathrm{d}L(\mu)}{\mathrm{d}t}=\{I_{i,k},L(\mu)\}=[L(\mu),A_{i,k}(\mu)] \tag{3.4}$$

where the second Lax matrix $A_{i,k}$ has the form

$$A_{i,k}(\mu)=\Phi_\lambda^{(i)}\operatorname{tr}_1(r_{21}(\lambda,\mu)L_1^{k-1}(\lambda)) \tag{3.5}$$

here trace tr_1 is taken over the first auxiliary space.

Let the representation space of the subalgebra $U_m(\mathfrak{g},\lambda,\mu,\dots,\nu)$ be an extended auxiliary space

$$V^{(m)}=\otimes_{i=1}^m V_i=V_1\otimes V_2\otimes\cdots\otimes V_m\qquad V_i\simeq\mathbb{C}^n. \tag{3.6}$$

In this case the elements $L_j(\lambda_j)$ (2.8) and $L^{(k)}_{j_1j_2\dots j_k}(\lambda,\mu,\dots,\nu)$ (2.9) belonging to the algebra $U_m(\mathfrak{g},\lambda,\mu,\dots,\nu)$ are the $n^m\times n^m$ matrices in $V^{(m)}$

$$\begin{gathered}L_j(\lambda_j)=I_1\otimes\cdots I_{j-1}\otimes L(\lambda_j)\otimes I_{j+1}\otimes\cdots\otimes I_m\\ L^{(k)}(\lambda,\mu,\dots,\nu)=\prod_{j=1}^m L_j(\lambda_j)\qquad \lambda_1=\lambda\qquad \lambda_2=\mu,\dots,\lambda_m=\nu\end{gathered} \tag{3.7}$$

here I means a $n\times n$ unit matrix and the subscript j shows in which of the spaces V_j in the whole space $V^{(m)}$ the matrix $L(\lambda)$ acts nontrivially.

The equation of evolution for the matrix $L^{(m)}$ has a commutator Lax form

$$\frac{\mathrm{d}}{\mathrm{d}t}L^{(m)}(\lambda,\mu\ldots,\nu)=[L^{(m)}(\lambda,\mu\ldots,\nu),A^{(m)}(\lambda,\mu\ldots,\nu)] \tag{3.8}$$

with the following second matrix

$$A^{(m)}(\lambda,\mu\ldots,\nu)=\sum_{j=1}^{m}A_j(\lambda_j)\qquad\lambda_1=\lambda\qquad\lambda_2=\mu,\ldots,\lambda_m=\nu \tag{3.9}$$

which is a sum of the matrices $A_j(\lambda_j)$ of the type (3.7) acting in the whole spaces $V^{(m)}$. The spectral invariants of the matrices $L^{(m)}$ give rise to a family of the integrals of motion in the involution as before.

Let us introduce matrices

$$L^{(m,\pi)}(\lambda,\mu,\ldots,\nu)=P_\pi L^{(m)}(\lambda,\mu,\ldots,\nu)=P_\pi L_1(\lambda)L_2(\mu)\ldots L_m(\nu). \tag{3.10}$$

Here the permutation matrix P_π in $V^{(m)}$ is determined by

$$P_\pi(x_1\otimes x_2\otimes\cdots\otimes x_m)=x_{\pi(1)}\otimes x_{\pi(2)}\otimes\cdots\otimes x_{\pi(m)} \tag{3.11}$$

for any set of vectors x_j in $\mathbb{C}^n$ or

$$P_\pi\cdot A_1B_2\ldots D_m=A_{\pi(1)}B_{\pi(2)}\ldots D_{\pi(m)}\cdot P_\pi\qquad P_\pi^2=I \tag{3.12}$$

for any $n\times n$ matrices $A,B,\ldots,D$ embedding in $V^{(m)}$ according to (3.7). The permutation of subscripts in (3.11) and (3.12) is defined by a certain Young diagram π [14].

For all the integrable systems the equations of evolution for the matrices $L^{(m,\pi)}$ are given by

$$\frac{\mathrm{d}}{\mathrm{d}t}L^{(m,\pi)}(\lambda,\mu\ldots,\nu)=L^{(m,\pi)}A^{(m)}(\lambda,\mu\ldots,\nu)-A^{(m,\pi)}(\lambda,\mu\ldots,\nu)L^{(m,\pi)}. \tag{3.13}$$

where matrix $A^{(m)}$ is given by (3.9) and the second matrix $A^{(m,\pi)}$ differs from it by the permutation of spectral parameters in accordance with the Young diagram π

$$\begin{aligned}A^{(m,\pi)}(\lambda,\mu\ldots,\nu)&=P_\pi A^{(m)}(\lambda,\mu\ldots,\nu)P_\pi^{-1}\\&=\sum_{j=1}^{m}A_j(\lambda_{\pi(j)})\qquad\lambda_1=\lambda\qquad\lambda_2=\mu,\ldots,\lambda_m=\nu.\end{aligned}$$

The right-hand side of (3.13) is a matrix commutator if and only if

$$A^{(m)}(\lambda,\mu\ldots,\nu)=A^{(m,\pi)}(\lambda,\mu\ldots,\nu)\Longleftrightarrow A(\lambda_j)=A(\lambda_{\pi(j)}). \tag{3.14}$$

It is easy to prove, this equation (3.14) is valid if either $A(\lambda)=A$ is independent on spectral parameters or all the spectral parameters at (3.14) are equal $\lambda=\mu=\cdots=\nu$.

Assuming (3.14) holds we can define new multivariable generating functions of the integrals of motion

$$\begin{aligned}&s_m^\pi(\lambda,\mu\ldots,\nu)=\frac{1}{m}\operatorname{tr}_{(m)}L^{(m,\pi)}(\lambda,\mu\ldots,\nu)\\&\frac{\mathrm{d}}{\mathrm{d}t}s_m^\pi(\lambda,\mu\ldots,\nu)=0.\end{aligned} \tag{3.15}$$

The trace $\operatorname{tr}_{(m)}$ in (3.15) is taken over the whole space $V^{(m)}$.

Now, from the theorem 1 one obtains the following.

Corollary 1. Superintegrable Hamiltonian systems on the loop algebra $\mathcal{L}(sl(n),\lambda)$ relate to a special point of the classical R-matrix.

It is essential that condition (3.14) is fulfilled for the independent on spectral parameter matrix A in (3.4)

$$A_{i,m} = \Phi_\lambda^{(i)} \operatorname{tr}_1(r_{21}(\lambda,\mu) L_1^{m-1}(\lambda)) = \text{constant} \neq 0.$$

We observe that matrix $A_{i,m}$ depends on spectral parameter μ via R-matrix only. Hence, the constant in the spectral sense matrix $A \equiv A(\mu)$ in theorem 1 and the corresponding special degenerate Hamiltonian are related to the singular points of the R-matrix.

As an example, the standard rational R-matrix for algebra $sl(n,\mathbb{C})$ is equal to

$$r_{12}(\lambda,\mu) = \frac{P_{12}}{\lambda-\mu}. \tag{3.16}$$

We can choose the special linear functional in (3.5) as residue at infinity

$$\Phi_\lambda(z) = -\operatorname{Res}|_{\lambda=\infty}(\phi(\lambda)\cdot z) \tag{3.17}$$

such that the second matrix

$$A(\mu) = -\operatorname{tr}_1\left[P \cdot \lim_{\lambda\to\infty} \frac{\phi(\lambda) L_1^{m-1}(\lambda)}{\lambda-\mu}\right] \tag{3.18}$$

is independent of spectral parameter μ. Moreover, if some invariant polynomial $\tau_k(\lambda)$ (3.1) has a nontrivial residue at $\lambda=\infty$ on the phase space

$$H = -\operatorname{Res}|_{\lambda=\infty}\phi(\lambda)\tau_k(\lambda) \tag{3.19}$$

which is chosen as a Hamiltonian, the corresponding Lax matrix A (3.18) does not equal zero. In this case the singular point $\lambda=\infty$ of R-matrix (3.16) is associated with the superintegrable Hamiltonian H (3.19) and with the new multivariable generating functions of the integrals of motion (3.15)

$$\{H, s_m^\pi(\lambda,\mu,\ldots,\nu)\} = 0. \tag{3.20}$$

For the algebra $sl(n)$ we lose the property of involution for these new multivariable functions and for the corresponding integrals of motion

$$\{s_m^{\pi_m}(\lambda_1,\lambda_2,\ldots,\lambda_m), s_k^{\pi_k}(\mu_1,\mu_2,\ldots,\mu_k)\} \neq 0. \tag{3.21}$$

These brackets are completely recovered by the polynomial R-brackets for the matrices $L^{(m,\pi)}$, as an example see (2.11) and (2.12).

Corollary 2. Complete set of integrals is determined by the generalized spectral surfaces

$$C(z,\lambda,\mu\ldots\nu) = \det(zI + L^{(m,\pi)}(\lambda,\mu\ldots\nu)) = 0 \qquad m \leqslant n \qquad \forall\pi. \tag{3.22}$$

At $\lambda_1 = \lambda_2 = \cdots = \lambda_m$ cross sections of these surfaces are equivalent to a usual spectral curve of $L(\lambda)$ determined by the characteristic equation

$$C(z,\lambda) = \det(zI + L(\lambda)) = 0 \tag{3.23}$$

which gives rise to integrals in the involution only.

The first proposition follows from the Lax form of the equation of motion. In additional, for any Hamiltonians $I_{i,k}$ (3.2) condition (3.14) is always fulfilled for the equivalent spectral parameters

$$\lambda = \lambda_j = \lambda_{\pi(j)} \qquad j = 1,\ldots,n \qquad \forall\pi \tag{3.24}$$

that corresponds to a choice of another basis of ad-invariant functions in the centre $\mathfrak{I}(g)$ [14] by using the outer powers of matrix $L(\lambda)$

$$s_m(\lambda) = \frac{1}{m}\operatorname{tr}_{(m)} L^{(m,\pi)}(\lambda,\lambda,\ldots,\lambda). \tag{3.25}$$

Symmetric functions $s_m(\lambda)$ can be expressed in the symmetric functions $\tau_m(\lambda)$ (3.1) according to Newton's formulae [14] and functions $s_m(\lambda)$ give rise to integrals in the involution.

Next let us consider Belavin R-matrices [2], which are meromorphic solutions of the classical Yang–Baxter equation such that

$$\begin{aligned} &r_{12}(\lambda, \mu) = r_{12}(\lambda - \mu) = -r_{21}(\mu - \lambda) \\ &r_{12}(\lambda - \mu) = \frac{P_{12}}{\lambda - \mu} + \mathrm{O}(1) \qquad P_{12} x \otimes y = y \otimes x \end{aligned} \tag{3.26}$$

that defines the expansion of r_{ij} in the neighbourhood of the singularity at $\lambda = \mu$. Here r_{ij} are rational, trigonometric or elliptic matrix functions on spectral parameters [2, 10, 21]. These matrices have a pole at $\lambda = \mu$ with a permutation operator P_{12} (3.26) as a residue. In addition, the rational function $r(\lambda - \mu)$ of two variables λ and μ has the special point at $\lambda = \infty$ in its domain of definition. In this point function $r(\lambda - \mu)$ has a distinct-from-zero residue, which is independent of the second spectral parameter μ. The generally accepted elliptic R-matrices [2, 10, 21] do not have such special points in their domains. Nevertheless, a construction similar to the rational case can be proposed as well.

The Lax equation (2.2) and R-bracket (2.3) are covariant under a similar transformation

$$L \to U^{-1} L U \qquad A \to U^{-1} A U \qquad r_{ij} \to U_1^{-1} U_2^{-1} r_{ij} U_1 U_2 \tag{3.27}$$

where U is a constant matrix on a phase space. If the matrix U depends on a spectral parameter, we can use similar transformations to construct additional poles of the R-matrix. In this case the multivariable generating functions $s_m^\pi(\lambda, \mu, \dots, \nu)$ are changed

$$s_m^\pi(\lambda, \mu, \dots, \nu) \to s_m^\pi(\lambda, \mu, \dots, \nu) = \frac{1}{m} \operatorname{tr}_{(m)}[Z_\pi L(\lambda) \otimes L(\mu) \dots \otimes L(\nu)] \tag{3.28}$$

here projector P_π in (3.20) is substituted by a matrix

$$Z_\pi = [U(\lambda) \otimes U(\mu) \dots \otimes U(\nu)]^{-1} \cdot P_\pi \cdot U(\lambda) \otimes U(\mu) \dots \otimes U(\nu) \tag{3.29}$$

depending on spectral parameters. For the equivalent spectral-parameter functions $s_m(\lambda)$ are covariant under similar transformations.

As an example, we study an elliptic R-matrix on the twisted-loop algebra $\mathcal{L}(sl(2), \sigma)$. Let us consider the periodic lattice $\Gamma = 2K\mathbb{Z} + 2\mathrm{i}K'\mathbb{Z}$, where K and K' are the standard elliptic integrals of the module $k \in [0, 1]$ and introduce the corresponding elliptic theta function $\Theta_{ij}(\lambda, k)$ as in [23]. In this notation the standard elliptic R-matrix is

$$r(\lambda - \mu) = \sum_{k=1}^{3} w_k(\lambda - \mu) \cdot \sigma_k \otimes \sigma_k \tag{3.30}$$

where σ_k are Pauli matrices and

$$\begin{aligned} &w_1(\lambda) = \frac{\Theta'_{11}(0, k)\Theta_{10}(\lambda, k)}{\Theta_{10}(0, k)\Theta_{11}(\lambda, k)} \qquad w_2(\lambda) = \frac{\Theta'_{11}(0, k)\Theta_{00}(\lambda, k)}{\Theta_{00}(0, k)\Theta_{11}(\lambda, k)} \\ &w_3(\lambda) = \frac{\Theta'_{11}(0, k)\Theta_{01}(\lambda, k)}{\Theta_{01}(0, k)\Theta_{11}(\lambda, k)}. \end{aligned}$$

The function $r(\lambda - \mu)$ (3.30) is meromorphic in $\mathbb{C}$ and has simple poles at $\lambda = \mu \bmod \Gamma$.

According to [23] we introduce the similar R-matrix

$$\begin{aligned} &\rho(\lambda, \mu) = U_{12}^{-1}(\lambda, \mu, \lambda_\infty) r(\lambda - \mu) U_{12}(\lambda, \mu, \lambda_\infty) \\ &U_{12}(\lambda, \mu, \lambda_\infty) = U_1(\lambda - \lambda_\infty + K) U_2(\mu - \lambda_\infty + K) \end{aligned}$$

with the following matrix $U(\lambda)$

$$U(\lambda)=\begin{pmatrix}\Theta_{01} & \Theta_{00}\\ -\Theta_{00} & -\Theta_{01}\end{pmatrix}(\xi,\tilde{k})\qquad \lambda=\frac{2\xi}{1+k}\qquad \tilde{k}=\frac{2\sqrt{k}}{1+k} \tag{3.31}$$

here λ_∞ is an arbitrary point. This R-matrix $\rho(\lambda,\mu)$ has been introduced for the purpose of separating the variables in [23]. More explicitly

$$\begin{aligned}\rho(\lambda,\mu)&=\sum_{j=1}^{3} w_j(\lambda-\mu,k)\cdot\sigma_j\otimes\sigma_j+[w_3(\mu-\lambda_\infty,k)-w_3(\lambda-\lambda_\infty,k)+w_0]\cdot\sigma_3\otimes\sigma_3\\ &\quad+\mathrm{i}w(\mu-\lambda_\infty,k)\cdot\sigma_3\otimes\sigma_2-\mathrm{i}w(\lambda-\lambda_\infty,k)\cdot\sigma_2\otimes\sigma_3\end{aligned} \tag{3.32}$$

$$w(\lambda,k)\equiv w_1(\lambda,k)=w_2(\lambda,k)=\frac{\Theta'_{11}}{\Theta_{10}}\frac{\Theta_{10}(\lambda,k)}{\Theta_{11}(\lambda,k)}\qquad w_3(\lambda)=\frac{\Theta'_{11}(\lambda,k)}{\Theta_{11}(\lambda,k)}$$

here w_0 is a some constant [23]. Let the poles of $\rho(\lambda,\mu)$ by the first spectral parameter λ be at the points

$$\lambda=\mu \mod\Gamma\qquad \mathrm{Res}|_{\lambda=\mu}\rho(\lambda,\mu)=P$$

$$\lambda=\lambda_\infty \mod\Gamma\qquad \mathrm{Res}|_{\lambda=\lambda_\infty}\rho(\lambda,\mu)=Z=(\sigma_3+\sigma_-+\sigma_+)\otimes\sigma_3.$$

So, if $L(\lambda)$ is an orbit of the new R-matrix $\rho(\lambda,\mu)$, such that the second matrix

$$A=\mathrm{tr}_1\left[Z\cdot\lim_{\lambda\to\lambda_\infty}\phi(\lambda)L(\lambda)\right]\neq 0$$

is a nontrivial constant in a spectral sense, the special Hamiltonian

$$H=\mathrm{Res}|_{\lambda=\lambda_\infty}\mathrm{tr}[\phi(\lambda)L^2(\lambda)]$$

is superintegrable.

In the next section we present some nontrivial examples of R-matrix orbits associated with superintegrable systems for the rational R-matrix.

4. Examples

The above construction of superintegrable systems can be applied to the Gaudin magnet, which was introduced in quantum mechanics [11]. The classical version turned out to be a useful example for developing a general group-theoretic approach to integrable system [10, 21].

We shall consider the rational Gaudin magnet related to $sl(N)$ algebra. The model in question is defined on the M coadjoint orbits of $sl(N)^*$ in variables $X_{ij}^{(m)}$, ($m=1,\ldots,M$, $i,j=1,\ldots,N$). The corresponding Lie–Poisson brackets are

$$\{X_{ij}^{(m)},X_{kl}^{(n)}\}=\delta_{mn}(X_{il}^{(m)}\delta_{jk}-X_{kj}^{(m)}\delta_{il}). \tag{4.1}$$

The coadjoint orbits are fixed by values $t_k^{(m)}$ of the ad-invariant functions on $sl(N)$

$$t_k^{(m)}=k^{-1}\,\mathrm{tr}(X^{(m)})^k\in\mathbb{C}. \tag{4.2}$$

The Poisson bracket (4.1) is nondegenerate on the manifold (4.2) having dimension $n=MN(N-1)/2$ for the case of generic orbits (all $t_i^{(m)}$ are distinct). In what follows we assume that the orbit is generic.

Fixing some element $Z\in sl(N)$ as a residue at infinity we consider the special Lax matrix $L(\lambda)\in\mathcal{L}(\oplus sl(N))$

$$L(\lambda)=Z+\sum_{m=1}^{M}\frac{X^{(m)}}{\lambda-\delta_m} \tag{4.3}$$

where $\{\delta_m\}$ is a set of M arbitrary constants. Matrix $L(\lambda)$ obeys the linear R-bracket (2.3) [10, 21] with the rational R-matrix (3.16).

The basis elements τ_k (3.1) are meromorphic functions of λ

$$\tau_k(\lambda) = \xi_k + \sum_{m=1}^{M}\sum_{j=1}^{k} \frac{I^j_{m,k}}{(\lambda-\delta_m)^j}$$

here $\xi_k = k^{-1}\,\mathrm{tr}\,Z^k$ and $I^k_{m,k} = t^{(m)}_k$ are fixed constants. Other residues $I^j_{m,k}$ form a family of $n = MN(N-1)/2$ independent integrals in the involution. It is immediately seen that the special Hamiltonians in this family

$$H^{(k)} = -\mathrm{Res}|_{\lambda=\infty}\tau_k(\lambda) = \sum_{m=1}^{M}\mathrm{Res}|_{\lambda=\delta_m}\tau_k(\lambda)$$

are nondegenerate functions on the generic coadjoint orbits of $sl(N)^*$ and they correspond to the constant in a spectral sense second Lax matrix (3.5)

$$A^{(k)} = Z^{k-1}.$$

So, Hamiltonians $H^{(k)}$ are superintegrable Hamiltonians and a complete set of the integrals of motion can be generated by (3.20). As an example, additional independent integrals of evolution may be constructed from the quantities

$$I^{\pi}_{m,k} = \mathrm{Res}_k s^{\pi}_m(\lambda,\mu,\ldots,\nu). \tag{4.4}$$

Here Res_k means the residue of some fixed order $k = (k_1, k_2, \ldots, k_m)$, $k_j \leqslant K_j$ at the points

$$\lambda = \delta_{j_1} \qquad \mu = \delta_{j_2}, \ldots, \nu = \delta_{j_m}$$

which belong to the divisor of the poles of a multivariable function $s^{\pi}_m(\lambda,\mu,\ldots,\nu)$

$$D = \{(\delta_j, K_j),\ j = 1,\ldots,M,\ (\infty, 1)\}.$$

As a second example, let us consider superintegrable natural systems on $\mathbb{R}^{2n}$ with the following Hamiltonians:

$$H = T + V = \sum a_{ij}p_ip_j + V(q_1,\ldots,q_n) \qquad a_{ij} \in \mathbb{R} \tag{4.5}$$

where $\{p_j, q_j\}^n_{j=1}$ are canonical variables. Several systems with the superintegrable Hamiltonians of the form (4.5) may be described by using 2×2 Lax matrix in the form

$$L(\lambda) = \begin{pmatrix} h & e \\ f & -h \end{pmatrix}(\lambda) \tag{4.6}$$

which satisfies the linear R-matrix algebra (2.3) with some matrix r_{ij}. Matrix $L(\lambda)$ has a single invariant polynomial (3.25)

$$s_2(\lambda) = \tfrac{1}{2}\,\mathrm{tr}[PL(\lambda)\otimes L(\lambda)] = \det L(\lambda) = -\tfrac{1}{2}\,\mathrm{tr}\,L^2(\lambda) = -\tfrac{1}{2}\tau_2(\lambda) \tag{4.7}$$

and one second-order complementary polynomial (3.20)

$$s_2(\lambda,\mu) = \tfrac{1}{2}\,\mathrm{tr}[PL(\lambda)\otimes L(\mu)] = h(\lambda)h(\mu) + \frac{e(\lambda)f(\mu)+f(\lambda)e(\mu)}{2}. \tag{4.8}$$

These polynomials will be the generating functions of the integrals of motion for the our superintegrable Hamiltonians (cf (1.5)).

If the potential V in (4.5) is equal to zero $V(q_1,\ldots,q_n) = 0$ we find geodesic motion, which is superintegrable. It is known [15], the Lax representation (4.6) associated with geodesic motion may be regarded as a generic point of the loop algebra $\mathcal{L}_D(sl(2))$ in a fundamental representation after an appropriate completion of $L(sl(2))$ associated with an arbitrary fixed divisor of poles [21]

$$D = \{(\delta_j, l_j),\ j = 1, \ldots, M, (\infty, K)\}.$$

Namely, according to [13], let us introduce function $e(\lambda)$ on the spectral parameter λ and on the flat coordinates q_j with its time derivative $e_x(\lambda) = \{H, e(\lambda)\}$, where H is a Hamiltonian of the geodesic motion (4.5) and the dependence of $e(\lambda)$ on the coordinates q_j is defined implicitly. Nevertheless, in the Lax equation

$$L_x(\lambda) = \{H, L\} = [L, A].$$

We define matrices L and A by

$$L(\lambda) = \begin{pmatrix} -e_x/2 & e \\ -e_{xx}/2 & e_x/2 \end{pmatrix}(\lambda) \qquad A(\lambda) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{4.9}$$

Below we prove that this Lax equation is associated with geodesic motion and the corresponding Hamiltonian H (by $V = 0$ at (4.5)) is equal to the highest residue at the distinguished Weierstrass point on the spectral curve of $L(\lambda)$ (3.23) at infinity $\lambda = \infty$.

To construct the Lax representation for a potential motion we can use the outer automorphism of the space of infinite-dimensional representations of underlying Lie algebra $sl(2)$ proposed in [25].

Applying this automorphism of the algebra $sl(2)$ directly to the Lax representation $L(\lambda)$ (4.6) on $L(sl(2))$ we obtain a family of the new Lax pairs

$$\begin{aligned} L'(\lambda) &= L(\lambda) - \sigma_- \cdot [\phi(\lambda)e^{-1}(\lambda)]_{MN} \\ A'(\lambda) &= A - \sigma_- \cdot [\phi(\lambda)e^{-2}(\lambda)]_{MN} = \begin{pmatrix} 0 & 1 \\ u_{MN}(\lambda) & 0 \end{pmatrix}. \end{aligned} \tag{4.10}$$

The corresponding Hamiltonian H' at the Lax equation (3.4)) remains to equal residue of the same order as for an initial geodesic motion, i.e. linear functional Φ_λ (3.2) is invariant under the mapping (4.10).

Function $\phi(\lambda)$ at (4.10) is an arbitrary function on spectral parameter and $[z]_{MN}$ means a restriction of z onto the ad^*_R-invariant Poisson subspace of the initial R-bracket [13, 25]. For the rational R-matrix (3.16) we can use the linear combinations of the following Laurent projections

$$[z]_{MN} = \left[\sum_{k=-\infty}^{+\infty} z_k\lambda^k\right]_{MN} \equiv \sum_{k=-M}^{N} z_k\lambda^k \tag{4.11}$$

or the Taylor projections by $M = 0$.

Mapping (4.10) plays the role of a dressing procedure allowing us to construct the Lax matrices $L'_{MN}(\lambda)$ for an infinite set of new integrable systems starting from the single known Lax matrix $L(\lambda)$ associated with one integrable model. The Lax matrix $L'(\lambda)$ (4.10) obeys the linear R-bracket (2.3), where constant r_{ij}-matrices are substituted by r'_{ij}-matrices depending on the dynamical variables

$$r_{12}(\lambda, \mu) \to r'_{12} = r_{12} - \frac{([\phi(\lambda)e^{-2}(\lambda)]_{MN} - [\phi(\mu)e^{-2}(\mu)]_{MN})}{(\lambda - \mu)} \cdot \sigma_- \otimes \sigma_-. \tag{4.12}$$

Associated with the matrices $L(\lambda)$ (4.9) and $L'(\lambda)$ (4.10) generating functions $s_2(\lambda)$ and $s'_2(\lambda)$ (4.7) obey the following equations of motion

$$\begin{aligned} \frac{\mathrm{d}s_2(\lambda)}{\mathrm{d}t} &= 0 \Rightarrow \partial_x^3 e(\lambda) = e_{xxx} = 0 \\ \frac{\mathrm{d}s'_2(\lambda)}{\mathrm{d}t} &= 0 \Rightarrow [\tfrac{1}{4}\partial_x^3 + u_{MN}(\lambda)\partial_x + \tfrac{1}{2}u_{MN,x}(\lambda)] \cdot e(\lambda) = 0. \end{aligned} \tag{4.13}$$

The integrated form of these relations (4.13) are

$$\begin{aligned} s_2(\lambda) &= \frac{e \cdot e_{xx}}{2} - \frac{e_x^2}{4} \\ s_2'(\lambda) &= \frac{e \cdot e_{xx}}{2} - \frac{e_x^2}{4} + e^2 \cdot u_{MN}(\lambda). \end{aligned} \tag{4.14}$$

A simple substitution for the entries of matrix $L(\lambda)$

$$\begin{aligned} &e(\lambda) = \mathcal{B}^2 \qquad h(\lambda) = -e_x/2 = -\mathcal{B}\mathcal{B}_x \\ &f(\lambda) = -e_{xx}/2 = -\mathcal{B}_x^2 - \mathcal{B}\mathcal{B}_{xx} \end{aligned} \tag{4.15}$$

turns determinants (4.14) into the form

$$s_2(\lambda) = \mathcal{B}^3\mathcal{B}_{xx} \qquad s_2'(\lambda) = \mathcal{B}^3\mathcal{B}_{xx} + \mathcal{B}^4\left[\frac{\phi(\lambda)}{\mathcal{B}^4}\right]_{MN} \tag{4.16}$$

if we use an explicit formula for the potential $u_{MN}(\lambda)$. These equations have the form of Newton's equations for the function $\mathcal{B}$

$$\begin{aligned} \mathcal{B}_{xx} &= s_2(\lambda)\mathcal{B}^{-3} \\ \mathcal{B}_{xx} &= s_2'(\lambda)\mathcal{B}^{-3} - \mathcal{B}\left[\frac{f(\lambda)}{\mathcal{B}^4}\right]_{MN}. \end{aligned} \tag{4.17}$$

To expand function $\mathcal{B}(\lambda)$ at a Laurent set, for example

$$\mathcal{B} = \sum_{j=0}^{N} q_{N-j}\lambda^j$$

it is easy to prove that the coefficients q_j obey Newton's equation of motion with a Hamiltonian of the form (4.5). Here we reinterpret the coefficients of $s_2(\lambda)$ and $s_2'(\lambda)$ in (4.17) not as functions on the phase space, but rather as integration constants. These expansions may be considered as an appropriate parametrization of the function $e(\lambda)$ in flat coordinates q_j. Note, in variables q_j mapping (4.10) affects only the potential (q-dependent) part of the integrals of motion I_k. The kinetic (momentum dependent) part of I_k remains unchanged. So, the dress mapping (4.10) allows us to change from a free motion on $\mathbb{R}^{2n}$ to a potential motion on $\mathbb{R}^{2n}$.

Namely, the function $e(\lambda)$ is given by

$$e(\lambda) = \sum_{i=1}^{K} e_i\lambda^i + \sum_{j=1}^{M}\sum_{k=1}^{l_j} \frac{e_{jk}}{(\lambda - \delta_j)^k} \tag{4.18}$$

and the Hamiltonian H associated with the Lax representation (4.9) or (4.10) is equal to a highest residue of $\det L(\lambda)$ at infinity $\lambda = \infty$

$$\Phi_\lambda(z) = \mathrm{Res}|_{\lambda=\infty}(\lambda^{-K}z). \tag{4.19}$$

Recall, that this functional defines Hamiltonians (3.2) for a geodesic motion and for all the potential motions simultaneously. Then the residues e_i and e_{jk} are easily restored in variables $\{q_j\}_{j=1}^n$ by using the function $\mathcal{B}$. As a first example, in the simple poles at $\lambda = \delta_j$, function $e(\lambda) = \mathcal{B}^2$ (4.18) has the following residues $e_{j1} = q_j^2$. Parametrization of the residues in the higher-order poles is discussed in [13]. As a second example, we consider the polynomial part of $e(\lambda)$ corresponding to a pole at infinity. Let

$$\mathcal{B}(\lambda) = \sum_{j=0}^{K} q_{K-j}\lambda^j \tag{4.20}$$

with $q_0 = 1$ and

$$e(\lambda) = \sum_{j=0}^{K} e_j \lambda^j \qquad h(\lambda) = \sum_{j=0}^{K} h_j \lambda^j \qquad f(\lambda) = \sum_{j=0}^{K} f_j \lambda^j \tag{4.21}$$

with $e_K = 1$, $h_K = 0$ and $f_K = 0$ due to the definition of A (4.9) and Φ_λ (4.19). Taking (4.15) and (4.18) into account we obtain

$$\begin{aligned} e_j &= \sum_{i=0}^{K-j} q_i q_{K-j-i} \qquad h_j = -\sum_{i=0}^{K-j} q_{i,x} q_{K-j-i} \\ f_j &= -\sum_{i=0}^{K-j} q_{i,x} q_{K-j-i,x} - \sum_{i=0}^{K-j} q_{i,xx} q_{K-j-i} \end{aligned} \tag{4.22}$$

where $q_x = \{H, q\}$ and we used the Newton formulae for a product of two sets. Canonically conjugate to the coordinates q_j momenta p_j can be derived from the R-matrix algebra (2.3) [13]. Some first polynomials are equal to

$$\begin{aligned} &K = 0 \qquad e(\lambda) = 1 \qquad h(\lambda) = 0 \\ &K = 1 \qquad e(\lambda) = \lambda + 2q_1 \qquad -h(\lambda) = p_1 \\ &K = 2 \qquad e(\lambda) = \lambda^2 + 2\lambda q_1 + (2q_2 + q_1^2) \\ &-h(\lambda) = \lambda p_2 + (p_1 + p_2 q_1) \\ &K = 3 \qquad e(\lambda) = \lambda^3 + 2\lambda^2 q_1 + \lambda(2q_2 + q_1^2) + 2(q_3 + q_1 q_2) \\ &-h(\lambda) = \lambda^2 p_3 + \lambda(p_2 + p_3 q_1) + (p_1 + p_2 q_1 + p_3 q_2). \end{aligned} \tag{4.23}$$

Note that the kinetic part of the Hamiltonian H (4.5) has a nondiagonal form in these variables

$$T = \sum_{j=1}^{K} p_j p_{K+1-j}. \tag{4.24}$$

Now we turn to the superintegrable systems. For the Taylor projections ($M = 0$) (4.11) the mapping (4.10) preserves the property of superintegrability if and only if $N \leqslant K$ according to (4.10) and (4.18). Here K is a highest order of a pole of the entry $e(\lambda)$ (4.18) at infinity and N is a highest power in projection (4.11). It is simpler to prove by using the definition of the corresponding Lax matrices (4.10), since in the definition of the R-matrices (4.12) one uses the square of the function $e(\lambda)$. In fact, mapping (4.10) preserves the nondynamical R-matrix (3.16), (4.12) and property of superintegrability simultaneously. By $N > K$ one obtains the dynamical R-matrix (4.12), which has the higher poles at $\lambda = \infty$ and the corresponding dynamical systems are no longer the superintegrable systems.

All these superintegrable systems are related to the special point of R-matrix and the associated second Lax matrix A' (4.10) remains a constant in a spectral sense under the mapping (4.10). All these superintegrable systems are related to the special stationary flows of the KdV hierarchy (4.13) and genus of the associated spectral curve of $L'(\lambda)$ determined by the characteristic equation (3.23) is no more than the number of degrees of freedom.

As an example, we present here several superintegrable systems. The entry $e(\lambda)$ has some simple poles at $\lambda = \delta_j$ and a distinguished pole at $\lambda = \infty$ of order K

$$e(\lambda) = P_K(\lambda) + \sum_{j=K+1}^{n} \frac{q_j^2}{\lambda - \delta_j}$$

where $P_K(\lambda)$ are polynomials given by (4.23). According to [25] the outer automorphism of the space of infinite-dimensional representations of the Lie algebra $sl(2)$ may be applied twice to the corresponding loop algebra. We can construct the Lax representations (4.10) with one free parameter $\phi(\lambda)$ and use the following realization of the underlying algebra $sl(2)$ with one free parameter β

$$s_3 = \frac{xp}{2} \qquad s_+ = \frac{x^2}{2} \qquad s_- = -\frac{p^2}{2} + \frac{f}{x^2} \qquad \Delta' = f. \tag{4.25}$$

Thus, for the some first values of $K = N$ the corresponding superintegrable potentials in (4.5) are

$$\begin{aligned}
K &= 0 \qquad V = \sum_{j=1}^{n} \left(q_j^2 + \frac{\beta_j}{q_j^2} \right) \\
K &= 1 \qquad V = 4q_1^2 + \sum_{j=2}^{n} \left(q_j^2 + \frac{\beta_j}{q_j^2} \right) \\
K &= 2 \qquad V = 4q_1^3 - 8q_1q_2 + \sum_{j=3}^{n} \left(q_j^2 + \frac{\beta_j}{q_j^2} \right) \\
K &= 3 \qquad V = -5q_1^4 + 12q_1^2q_2 - 4q_2^2 - 8q_1q_3 + \sum_{j=4}^{n} \left(q_j^2 + \frac{\beta_j}{q_j^2} \right).
\end{aligned}$$

If $N = K$, then the dress mapping (4.10) has the $N + 1$ arbitrary parameters given by the function $\phi(\lambda) = \sum_{j=0}^{N} \alpha_j \lambda^j$. These parameters are related to the canonical shifts of variables $q_j \to q_j + \alpha_j$ and to the common rescaling $V \to \alpha_N V$ in the presented potentials. Additional integrals of motion may be constructed by the rule (4.4) from the multivariable generating function $s_2(\lambda, \mu)$ (4.8).

We may construct similar superintegrable systems with rational potentials by using a general form of the entry $e(\lambda)$ (4.18) with the higher-order poles and applying a more general Laurent projection (4.11). The presented method can be employed to construct superintegrable systems on the other Riemannian spaces of constant curvature [15]. The corresponding quantum systems may be obtained by canonical quantization [7, 25].

Taking into account the R-bracket (2.3) one can conclude that the entries $e(\lambda)$ and $f(\lambda)$ could play roles similar to the standard creation and annihilation operators for harmonic oscillator [14, 9]. By using a similar transformation of matrices $L(\lambda)$ or $L'(\lambda)$ one can obtain the symmetric representation of these matrices

$$e(\lambda) = e(\lambda, a_1, a_2, \ldots, a_n) \qquad f(\lambda) = e^+(\lambda) \qquad h(\lambda) = h^+(\lambda).$$

Here a_j, a_j^+ are the standard creation and annihilation operators. In this symmetric representation of the Lax matrix the usual method of spectrum-generating algebras [4, 18] is a part of the standard Bethe ansatz [14, 9]. It should be emphasized, that the algebraic Bethe ansatz is a sufficiently universal procedure, which slightly depends on the particular system in question. It allows us to interpret various concrete models as some representations of a single generalized model, which is defined by its R-matrix only.

As a third example, let us consider the Calogero–Moser systems. It is well known [20], that both the Toda models and the Calogero–Moser models are obtained by Hamiltonian reduction of the geodesic motion on the cotangent bundle T^*G of a Lie group G. For the geodesic motion on symmetric spaces of zero curvature the canonical 2-form, the free

Hamiltonian and equations of motion are equal to

$$\begin{aligned} &w = \mathrm{tr}(\mathrm{d}y \wedge \mathrm{d}x) \qquad H = \tfrac{1}{2}\,\mathrm{tr}(y^2) \\ &\dot{x} = y \qquad \dot{y} = 0. \end{aligned} \tag{4.26}$$

For geodesic motion on symmetric spaces of positive or negative curvature these quantities read

$$\begin{aligned} &w = \mathrm{tr}(x^{-1}\,\mathrm{d}y \wedge x^{-1}\,\mathrm{d}x) \qquad H = \tfrac{1}{2}\,\mathrm{tr}(yx^{-1})^2 \\ &\dot{x} = y \qquad \dot{y} = yx^{-1}y. \end{aligned} \tag{4.27}$$

The Hamiltonians (4.26) and (4.27) have the following sets of integrals in the involution

$$I_k = \mathrm{tr}(y^k) \qquad \text{and} \qquad I_k = \mathrm{tr}(yx^{-1})^k.$$

The additional integrals—'projections of angular momentum' (1.5)—are equal to

$$I_{jk} = \mathrm{tr}(qp^{j-1})\,\mathrm{tr}(p^k) - \mathrm{tr}(p^j)\,\mathrm{tr}(qp^{k-1}). \tag{4.28}$$

Here $q = x$ and $p = y$ for the first equations of geodesic motion (4.26) and $q = \ln x$ or $q = \ln y$ with $p = yx^{-1}$ for the second equations of geodesic motion (4.27).

In the reduction process the Lax matrices of the reduced system are expressed in terms of x by a formula of the type $L = zxz^{-1}$, where z is some element in G [20]. For the geodesic motion (4.26) associated with the Calogero model with the rational and trigonometric potentials, the Hamiltonian H (4.26) remains superintegrable and images of integrals (4.28) are integrals of a reduced system [26]. In the quantum mechanics whole polynomial algebra of the integrals of motion for the Calogero model is introduced in [16].

Let us show how the image of a superintegrable Hamiltonian (4.26) appears in the R-matrix formalism associated with the Calogero model. For instance, consider the Euler–Calogero–Moser system [26]. Introduce a set of dynamical variables $\{(q_j, p_j)\}_{j=1}^N$ and $\{f_{ij}\}_{i,j=1}^N$ $(f_{ij} = -f_{ji})$ together with the Poisson brackets

$$\{p_j, q_k\} = \delta_{jk} \tag{4.29}$$

$$\{f_{ij}, f_{kl}\} = \tfrac{1}{2}(\delta_{il} f_{jk} + \delta_{ki} f_{lj} + \delta_{jk} f_{il} + \delta_{lj} f_{ki}). \tag{4.30}$$

In order to have a nondegenerate Poisson bracket it is assumed that the variables f_{ij} are restricted to a symplectic submanifold of (4.30). The Hamiltonian and the Lax matrix for the Euler–Calogero–Moser system [26] are given by

$$\begin{aligned} H &= \frac{1}{2}\sum_{j=1}^N p_j^2 + \frac{1}{2}\sum_{i,j=1\ i\neq j}^N \frac{f_{ij}^2}{(q_i - q_j)^2} \\ L(\lambda) &= \sum_{j=1}^N p_j e_{jj} + \sum_{i,j=1\ i\neq j}^N \left(\frac{1}{q_i - q_j} + \frac{1}{\lambda}\right) f_{ij} e_{ij} \end{aligned} \tag{4.31}$$

with the corresponding R-matrix in the form [3]

$$\begin{aligned} r_{12}(\lambda, \mu) = &-\frac{\lambda}{\lambda^2 - \mu^2}\sum_{j=1}^N e_{jj} \otimes e_{jj} - \frac{1}{2}\sum_{i,j=1\ i\neq j}^N \left(\frac{1}{q_i - q_j} + \frac{1}{\lambda + \mu}\right) e_{ij} \otimes e_{ij} \\ &-\frac{1}{2}\sum_{i,j=1\ i\neq j}^N \left(\frac{1}{q_i - q_j} + \frac{1}{\lambda - \mu}\right) e_{ij} \otimes e_{ji} \end{aligned} \tag{4.32}$$

where $(e_{ij})_{kl} = \delta_{ik}\delta_{jl}$. In the reduction process this R-matrix inherits the singular point $\lambda = \infty$ from the initial rational R-matrix. The corresponding superintegrable Hamiltonian (4.31) may be defined by (3.2) with $\phi = \frac{1}{2} \cdot \lambda^{-1}$

$$H = \Phi_\lambda[\mathrm{tr}\, L^2(\lambda)] = \tfrac{1}{2}\mathrm{Res}|_{\lambda=\infty}[\lambda^{-1} \cdot \mathrm{tr}\, L^2(\lambda)].$$

and the second Lax matrix is independent of spectral parameter

$$A = \Phi_\lambda\, \mathrm{tr}_1[r_{21}(\lambda, \mu)L_1(\lambda)] = \sum_{i,j=1\ i\neq j}^{N} \frac{f_{ij}}{(q_i - q_j)^2} e_{ij}.$$

The higher flows with $\phi(\lambda) = 1/k \cdot \lambda^{-k-1}$ in (3.2) are superintegrable as well [26].

5. Conclusions

We have seen that superintegrable systems connected to geodesic motion can be realized as isospectral flows on coadjoint orbits of loop algebras in the framework of R-matrix formalism. All these systems are associated with the special singular point of the classical R-matrix.

Another classical superintegrable system with an arbitrary number of degrees of freedom is the Kepler problem [1, 20]. In the proposed scheme we can consider a free geodesic motion on the momentum sphere and use the stereographic projection with an appropriate change of the time variable to study the Kepler problem [19]. However, this transformation could violate the R-bracket (2.3) for the corresponding Lax matrix. It would be interesting to construct the 2×2 Lax matrix for the Kepler problem and for the Kepler-like superintegrable potentials listed in [8].

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